

Transitivity of codimension one non-invertible conservative skew-products

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Abstract

In this work we explore the problem of transitivity of volume preserving skew-products endomorphisms of the n -torus. More specifically, we establish relationships between transitivity and the action induced by the skew-product in the fundamental group.

1 Introduction

In dynamical systems, an important family to study is the family of skew-products. They are easy to build and have a simple structure, yet they have enough complexity to model more general systems. Our focus in this paper will be volume-preserving non-invertible skew-products. A general goal for volume-preserving maps is to know whether or not they are ergodic. Since ergodicity is stronger than transitivity, we consider a good starting point to address the transitivity.

By a toral endomorphism we mean a surjective local homeomorphism $f : \mathbb{T}^n \rightarrow \mathbb{T}^n$. In other words, a covering map from \mathbb{T}^n to itself. Let μ be the Haar measure on \mathbb{T}^n . We say that f is *volume-preserving* (or *conservative*) if $\mu(f^{-1}(B)) = \mu(B)$ for every Borel measurable set $B \subset \mathbb{T}^n$.

We say that f is transitive if there exists $z \in \mathbb{T}^n$ such that $\mathbb{T}^n = \overline{\{f^n(z) : n \in \mathbb{N}\}}$.

It is reasonable to expect transitivity for volume-preserving non-invertible endomorphisms under quite general circumstances. First of all, note that linear (hence volume-preserving) non-invertible toral endomorphisms are always transitive (in fact ergodic [AH]). Indeed, they are robustly transitive: every C^1 close endomorphism (not necessarily conservative) is also transitive. In dimension two, every conservative endomorphism homotopic to a non-invertible hyperbolic linear map is transitive [A]. Furthermore, Lizana and Pujals in [LP] provided sufficient conditions for C^1 endomorphisms to be robustly transitive. Rather than dealing with conservative endomorphism, they consider endomorphisms with Jacobian larger than one.

Given $h : \mathbb{T}^{n-1} \rightarrow \mathbb{T}^{n-1}$ and $g : \mathbb{T}^{n-1} \times \mathbb{T}^1 \rightarrow \mathbb{T}^1$ we define $f : \mathbb{T}^n \rightarrow \mathbb{T}^n$ by $f(x, t) = (h(x), g(x, t)) \forall x \in \mathbb{T}^{n-1}, \forall t \in \mathbb{T}^1$. We say that f is a skew-product of codimension 1 and has the form $f = (h, g)$. We shall refer to h as the action in the base and g as the action in the fibers. For $x \in \mathbb{T}^{n-1}$ let us define the map $g_x : \mathbb{T}^1 \rightarrow \mathbb{T}^1$ by $g_x(t) = g(x, t)$. Note that, since f is a covering map, so are h and g_x for every $x \in \mathbb{T}^{n-1}$. Let $\deg(f)$, $\deg(h)$, and $\deg(g_x)$ denote their (unsigned) degrees, this is, the number of preimages of any point. Since

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g is continuous and $\deg(g_x)$ is a homotopy invariant, the number $\deg(g_x)$ does not depend on x and we denote it by $\deg(g)$. Observe that $\deg(f) = \deg(g)\deg(h)$.

A classical family of invertible skew-products are maps $f : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ of the form $f(x, t) = (x + \alpha, t + \phi(x))$, where $\alpha \in \mathbb{R}$ is an irrational number and $\phi : \mathbb{T}^1 \rightarrow \mathbb{T}^1$ is a continuous map. If the cohomological equation

$$u(x + \alpha) - u(x) = \phi(x),$$

has a continuous solution, then f is conjugated to $f_0(x, t) = (x + \alpha, t)$ and therefore it is not transitive. This is an example of a non transitive skew-product where $\deg(f) = \deg(h) = \deg(g) = 1$. Similar examples can be constructed by replacing the base map $x \mapsto x + \alpha$ by $x \mapsto kx \mod 1$. In this case we have $\deg(f) = \deg(h) = |k|$ and $\deg(g) = 1$.

We would like to address now what happens when $\deg(g) \geq 2$. Observe that if $f = (h, g)$ preserves the Haar measure on \mathbb{T}^n , then h preserves the Haar measure on \mathbb{T}^{n-1} . Moreover, if f is transitive, so is h .

In order to announce the main theorem of this article we will need to define a linear map associated to a torus endomorphism. Given $f : M \rightarrow M$, let $f_\# : \pi_1(M) \rightarrow \pi_1(M)$ be the induced morphism on the fundamental group $\pi_1(M)$ of M . If we take $M = \mathbb{T}^n$, then $\pi_1(\mathbb{T}^n)$ is isomorphic to \mathbb{Z}^n and we can represent $f_\#$ by a linear matrix $A_f \in M_n(\mathbb{Z})$. We shall often refer to the matrix A_f (or the maps it defines on \mathbb{R}^n and \mathbb{T}^n) as the *linear part of f* . Observe that if f is a skew-product, we have $f(\{x\} \times \mathbb{T}^1) = \{h(x)\} \times \mathbb{T}^1$ for every $x \in \mathbb{T}^{n-1}$. Therefore the vector $e_n = (0, \dots, 0, 1)$ is an eigenvector of A_f . It is not hard to see that the eigenvalue associated to e_n is either $\deg(g)$ or $-\deg(g)$. For A_f , we consider its Jordan normal form J and the Jordan block J_n associated to the eigenvector e_n .

The main Theorem is the following:

Theorem 1: *Let $f : \mathbb{T}^n \rightarrow \mathbb{T}^n$ be a skew-product endomorphism of codimension 1 of the form $f = (h, g)$, with $h : \mathbb{T}^{n-1} \rightarrow \mathbb{T}^{n-1}$ transitive and $\deg(g) \geq 2$. If f is volume-preserving and $\dim(J_n) = 1$, then f is transitive.*

We emphasize that our result is purely topological, this is, it does not rely on any C^r regularity of the maps f , h and g and, in particular, does not make use of any hyperbolic structure. Neither do our proofs require the density (not even the existence) of periodic points.

Before discussing the hypothesis $\dim(J_n) = 1$ in more detail we would like to point out some particular cases in which it holds. Suppose that $f = (h, g)$ is a volume-preserving skew-product with h transitive, $\deg(g) \geq 2$ and $\deg(h) = 1$. The hypothesis on the degree of h means that it is a homeomorphism, and by a previous observation, a volume-preserving homeomorphism. It is not hard to see then, that at least in the C^1 case, each of the maps $g_x : \mathbb{T}^1 \rightarrow \mathbb{T}^1$ is uniformly expanding. (In the C^0 case a variant of uniform expansion occurs.) Assuming transitivity of h allows us to easily conclude that f itself is transitive. This proof is unrelated to (and indeed much easier than) the proof of Theorem 1. We therefore state it separately:

Theorem 2: *Let $f : \mathbb{T}^n \rightarrow \mathbb{T}^n$ be a skew-product of codimension 1 of the form $f = (h, g)$. If f is a volume-preserving endomorphism, h is a transitive homeomorphism, and $|\deg(g)| \geq 2$, then f is transitive.*

At a first glance, one could imagine that it would be easier to obtain transitivity in the case where $\deg(h) \geq 2$, due to the extra complexity coming from the base. But that is not the case, because when $\deg(h) \geq 2$, the condition of f being volume-preserving does not imply that the g_x have to be uniformly expanding.

Theorem 3: *Given $n \geq 2$, and $k \geq 2$ there exists a volume-preserving skew-product endomorphism $f : \mathbb{T}^n \rightarrow \mathbb{T}^n$ of codimension 1 of the form $f = (h, g)$ with $\deg(h) \geq 2$ and $\deg(g) = k$ such that*

- *h is transitive,*
- *there are a fixed point x_0 of h , and an interval $I \subset \mathbb{T}^1$ such that g_{x_0} is uniformly contracting on I ,*
- *the linear part of f is given by the matrix*

$$A_f = \begin{pmatrix} & & & 0 \\ & 2Id & & \vdots \\ & & & 0 \\ 1 & \dots & 1 & k \end{pmatrix},$$

where k is the eigenvalue associated to e_n .

Note that if we take $k > 2$, then the form of the linear part of f in Theorem 3 implies that f satisfies the hypothesis of Theorem 1 and is therefore transitive.

Theorem 3 suggests that, in order to deal with transitivity in the case where $|\deg(h)| \geq 2$, one has to adopt global arguments that make use of the way that f wraps curves around the manifold rather than localized behavior such as expansion or contraction near a given point.

Let us now give some examples where the hypothesis $\dim(J_n) = 1$ holds, obtaining some corollaries of Theorem 1:

Corollary 1: *Let $f : \mathbb{T}^n \rightarrow \mathbb{T}^n$ be a skew-product of codimension 1 with the form $f = (h, g)$. Suppose that f is a volume-preserving endomorphism, h is a transitive endomorphism, and $1 \leq \deg(h) < \deg(g)$. Then, f is transitive.*

This implies that Theorem 2 is really a corollary of Theorem 1. The result is also true for another type of domination:

Corollary 2: *Let $f : \mathbb{T}^n \rightarrow \mathbb{T}^n$ be a skew-product of codimension 1 with the form $f = (h, g)$. If f is a volume-preserving endomorphism, h is a transitive endomorphism, $\deg(g) > 1$ and $|A_h v| > \deg(g)|v| \ \forall v \in \mathbb{R}^n - \{0\}$, then f is transitive.*

Finally if A_f is diagonalizable, then all the Jordan blocks have dimension 1 and therefore we have:

Corollary 3: *Let $f : \mathbb{T}^n \rightarrow \mathbb{T}^n$ be a skew-product of codimension 1 with the form $f = (h, g)$. If f is a volume-preserving endomorphism, h is a transitive endomorphism, $\deg(g) > 1$, and A_f is diagonalizable, then f is transitive.*

The examples built in Theorem 3 can verify the hypothesis of Theorem 1 or the previous corollaries and therefore they would still be transitive.

Let us give a sketch of the proof of Theorem 1:

We call an invariant region an open set which verifies $f^{-1}(U) = U$. If f is a volume-preserving endomorphism, the lack of transitivity is equivalent to the existence of more than one invariant region (Check Proposition 3.1). We start by studying the structure of the fundamental group of such invariant regions. Our starting point is the set of techniques used in [A], where the first author proved that, given a volume-preserving toral endomorphism $f : \mathbb{T}^2 \rightarrow \mathbb{T}^2$, with $|\deg(f)| \geq 2$, such that A_f is hyperbolic, then f is transitive.

Using non-invertibility and the hypothesis that f is conservative one can prove that, if $i : U \rightarrow \mathbb{T}^n$ denotes the inclusion, then $i_*(\pi_1(U))$ is not trivial. This is the main use we give to the volume-preserving hypothesis and the same results could be obtained by switching for

the hypothesis $\Omega(f) = \mathbb{T}^n$ (where $\Omega(f)$ is the non-wandering set) as it is done by Ranter in [R]. Our next step is to conclude that, not only is $i_{\#}(\pi_1(U))$ non-trivial, but it has to be big enough such that the action of $f_{\#}|_{i_{\#}(\pi_1(U))}$ has the same “degree” as $f_{\#}$. After that, we take a lift from f and using the skew-product structure we construct an invariant hypersurface S . This hypersurface is obtained by the expansiveness of the linear part of f along the fibers and the $\dim(J_n) = 1$ hypothesis. In particular, the dynamic of S is conjugated to the dynamic of h . By the previous arguments we prove that the lift of any invariant region intersects such hyper-surface and from the transitivity of h we obtain a contradiction.

Let us observe that the hypothesis $\dim(J_n) = 1$ is a necessary condition to imply the existence of the hypersurface and therefore essential to our proof, yet we do not know whether there exists a counter-example to Theorem 1 if this hypothesis is removed.

In section 2 we proof Theorem 2 and Theorem 3. In section 3 we develop the setting we will be working and prove the results from algebraic topology we will need. In section 4 we study some properties of invariant subspaces from integer matrices. In section 5 we prove Theorem 1. Observe that all the results stated in section 3 hold for toral endomorphisms, not just skew-products.

2 Theorem 2 and Theorem 3

Let us see the proof of Theorem 2

Proof. Let ν be the Haar measure on \mathbb{T}^{n-1} and let λ be the Haar measure on \mathbb{T} . We shall first show that if f preserves μ , then h preserves ν . Let $r_1 : \mathbb{T}^n \rightarrow \mathbb{T}^{n-1}$ be the projection $r_1(x_1, \dots, x_{n-1}, x_n) = (x_1, \dots, x_{n-1}) \forall (x_1, \dots, x_n) \in \mathbb{T}^n$. Given $B \subset \mathbb{T}^{n-1}$, we have that $r_1^{-1}(B) = B \times \mathbb{T}^1$. Thus $r_{1*}\mu = \nu$. By the skew-product structure $f^{-1}(B \times \mathbb{T}^1) = h^{-1}(B) \times \mathbb{T}^1$. Since f is volume-preserving:

$$\nu(h^{-1}(B)) = \mu(h^{-1}(B) \times \mathbb{T}) = \mu(f^{-1}(B \times \mathbb{T}^1)) = \mu(B \times \mathbb{T}^1) = \nu(B).$$

The next step is to show that if h is a homeomorphism, then g_x preserves λ for every $x \in \mathbb{T}^{n-1}$. It is instructive to consider the case in which f is of class C^1 . In this case, f , h , and each of the g_x have well defined Jacobians. Let us denote these by $J(f, \cdot)$, $J(h, \cdot)$, and $J(g_x, \cdot)$ respectively. Since f preserves μ , we must have

$$1 = \sum_{(y,s) \in f^{-1}(x,t)} \frac{1}{J(f, (y,s))} = \sum_{y \in h^{-1}(x)} \frac{1}{J(h, y)} \sum_{s \in g_y^{-1}(t)} \frac{1}{J(g_y, s)} \quad \forall x \in \mathbb{T}^{n-1}, \forall t \in \mathbb{T}^1.$$

Since h is a homeomorphism, $\#h^{-1}(x) = 1$ and since it is volume-preserving, $J(h, y) = 1$. If we combine this with the previous equation, we obtain that:

$$\sum_{s \in g_y^{-1}(t)} \frac{1}{J(g_y, s)} = 1,$$

where $y = h^{-1}(x)$. Since $|\deg(g)| \geq 2$ and $J(g_y, s) > 0$, we can conclude that $J(g_y, s) > 1 \forall y \in \mathbb{T}^{n-1}, \forall s \in \mathbb{T}^1$. By continuity of dg and compactness of \mathbb{T}^n , we have that $J(g_y, s) > 1 + \epsilon$ for some $\epsilon > 0$. This implies that g is expanding in the fibers.

In particular, given $I \subset \mathbb{T}^1$ there exists $k = k(I) > 0$ such that for all $x \in \mathbb{T}^{n-1}$, $f^n(\{x\} \times I) = \{h^k(x)\} \times \mathbb{T}^1$. Let us take U_1 and U_2 open neighborhoods of \mathbb{T}^{n-1} , and I_1 and I_2 open neighborhoods of \mathbb{T}^1 . We want to prove that $f^{k_1}(U_1 \times I_1) \cap U_2 \times I_2 \neq \emptyset$ for some $k_1 > 0$. Taking k associated to I_1 and using the transitivity of h , there exists $k_1 > k$ such that $h^{k_1}(U_1) \cap V_1 \neq \emptyset$. Then, $(h^{k_1}(U_1) \cap V_1) \times I_2 \subset f^{k_1}(U_1 \times I_1) \cap U_2 \times I_2$.

Now let us consider the more general case in which f is only assumed to be a continuous surjective local homeomorphism. Our first assertion is that each g_x preserves λ .

For the purpose of contradiction, suppose there is some x such that g_x does not preserve λ . That is equivalent to say that there is some continuous function $\phi : \mathbb{T} \rightarrow \mathbb{R}$ such that

$$\int \phi \circ g_x \, d\lambda < \int \phi \, d\lambda. \quad (1)$$

Since the map $\mathbb{T}^{n-1} \ni x \mapsto g_x \in C^0(\mathbb{T}, \mathbb{T})$ is continuous, if (1) holds for some x , then it holds in an open set $U \subset \mathbb{T}^{n-1}$. Let $\psi : \mathbb{T}^{n-1} \rightarrow \mathbb{R}$ be a non-negative continuous function, supported in U , such that $\int \psi \, d\nu > 0$, and let $\varphi : \mathbb{T}^n \rightarrow \mathbb{R}$ be defined by $\varphi(x, t) = \phi(x)\psi(t)$. We claim that $\int \varphi \circ f \, d\mu < \int \varphi \, d\mu$, contradicting the f -invariance of μ .

Indeed,

$$\int \varphi \circ f \, d\mu = \int \left(\int \phi(h(x)) \psi(g_x(t)) \, d\lambda(t) \right) d\nu(x) \quad (2)$$

$$= \int \phi(h(x)) \left(\int \psi \circ g_x \, d\lambda \right) d\nu(x) \quad (3)$$

$$< \int \phi \circ h \, d\mu \int \psi \, d\lambda \quad (4)$$

$$= \int \phi \, d\nu \int \psi \, d\lambda = \int \varphi \, d\mu, \quad (5)$$

and we have arrived at the desired contradiction.

Now, since g_x is not (necessarily) of class C^1 , there may not exist $\epsilon > 0$ such that $\lambda(g_x(I)) > (1+\epsilon)\lambda(I)$ for every interval $I \subset \mathbb{T}$ such that $g_x : I \rightarrow g_x(I)$ is a homeomorphism. However, $\lambda(g_x(I))$ is always larger than $\lambda(I)$ so, by compactness, given any $K > 0$ there is a $\delta > 0$ such that if $\lambda(I) \geq K$ and g_x is a homeomorphism from I onto its image, then $\lambda(g_x(I)) \geq \lambda(I) + \delta$. Writing $I_n = g_{h^{n-1}(x)} \circ \dots \circ g_{h(x)} \circ g_x(I)$, $n \geq 0$, then we see by induction that $\lambda(I_k) \geq \lambda(I) + k\delta$ as long as $g_{h^k(x)} : I_k \rightarrow I_{k+1}$ is a homeomorphism. But $\lambda(I) + k\delta$ is larger than 1 for k sufficiently large, so there must be some k such that $I_k = \mathbb{T}$. Now we may apply the same argument as in the C^1 case to conclude that f is transitive. \square

Observe that when $|\deg(h)| \geq 2$, we no longer have the condition $J(h, y) = 1$ in the C^1 case. Instead, it is replaced by $\sum_{y \in h^{-1}(x)} \frac{1}{J(h, y)} = 1$. Since the sum has more than one term, this will imply that $J(h, x) > 1$ for every $x \in \mathbb{T}^{n-1}$, this is that h expands volume on sufficiently small sets. But $J(h, \cdot)$ does not have to be constant, since a lesser volume expansion on some point x_1 can be compensated by a greater volume expansion on a point x_2 , where x_1 and x_2 have the same image under h . This flexibility makes it possible to have a volume-preserving skew product which is contracting on some of its fibers. This is the content of Theorem 3. In particular, proving Theorem 1 will require an entirely different approach than that in Theorem 2.

Proof of Theorem 3. The example we are going to build will be piecewise linear and therefore C^1 in an open and dense set with full measure. The volume-preserving property will then be guaranteed by making sure that the equation

$$\sum_{(x,t) \in f^{-1}(y,s)} \frac{1}{J(f, (x,t))} = 1 \quad (6)$$

hold for almost every point (x, t) in $\mathbb{T}^{n-1} \times \mathbb{T}$.

Let $m = n - 1$ denote the dimension of the base. For the base map $h : \mathbb{T}^m \rightarrow \mathbb{T}^m$ we take the linear endomorphism induced by the matrix $A = 2 \cdot Id$, where Id is the identity matrix of size $m \times m$. Note that $\deg(h) = |\det(A)| = 2^m$. By standard arguments, h is transitive.

The action in the fibers will have two degrees of freedom in its construction. The first one is going to be the degree, denoted by $k = \deg(g) \geq 2$. The second one is going to be the rate of contraction $\lambda \in (0, 1)$. In our construction we are going to need $\lambda \in (1/2, 1)$. We define the map $\phi : \mathbb{T}^1 \rightarrow \mathbb{T}^1$ by

$$\phi(t) = \begin{cases} \lambda t & \text{if } t \in [0, 1/(2\lambda)] \\ \frac{(2k-1)\lambda}{2\lambda-1}(t - 1/(2\lambda)) + 1/2 \mod 1 & \text{if } t \in [1/(2\lambda), 1] \end{cases}$$

Let us observe the following:

- $\lambda > 1/2$, so $2\lambda - 1 > 1$.
- ϕ is clearly continuous at $1/(2\lambda)$ and, to check the continuity at 0, observe that $\frac{(2k-1)\lambda}{2\lambda-1}(1 - 1/(2\lambda)) + 1/2 = k \in \mathbb{N}$.
- ϕ contracts by the rate λ the interval $[0, 1/(2\lambda)]$ and expands by the rate $\eta = \frac{(2k-1)\lambda}{2\lambda-1}$ the interval $[1/(2\lambda), 1]$.

We define $g : \mathbb{T}^m \times \mathbb{T}^1 \rightarrow \mathbb{T}^1$ by $g(x_1, \dots, x_m, t) = x_1 + \dots + x_m + \phi(t + 1/(4\lambda)) - 1/4$ and finally $f : \mathbb{T}^n \rightarrow \mathbb{T}^n$ as the skew-product of the form (h, g) .

In the definition of g , the addition and the subtraction of the constants $1/(4\lambda)$ and $1/4$ are to obtain $f(0) = 0$ and $\frac{\partial}{\partial t}g(x, t) = \lambda$ in a neighborhood $B \times I \subset \mathbb{T}^n$ of 0. From this we can conclude all the desired properties in the statement of Theorem 3, except that f is conservative.

Given $a \in \mathbb{T}^1$, denote by $\psi_a : \mathbb{T}^1 \rightarrow \mathbb{T}^1$ the map $\psi_a(t) = a + \phi(t + 1/(4\lambda)) - 1/4$.

in order to prove that f is conservative we need to understand the distribution of the preimages of a point. Given $(y, s) \in \mathbb{T}^n$, we have that

$$f^{-1}(y, s) = \bigcup_{x \in h^{-1}(y)} \{(x, t) \in \mathbb{T}^n : g(x, t) = s\}.$$

As we said before $h^{-1}(y)$ has 2^m points. Fix $y_0 \in p^{-1}(h^{-1}(y))$ and let $X_0 = \{y_0 + \frac{a_1}{2}e_1 + \dots + \frac{a_n}{2}e_n \in \mathbb{R}^n : a_i \in \{0, 1\}\}$ where e_1, \dots, e_n is the canonical basis of \mathbb{R}^n . Then, the natural projection $p : \mathbb{R}^n \rightarrow \mathbb{T}^n$ restricted to X_0 is a bijection onto $h^{-1}(y)$. Given $x \in h^{-1}(y)$, take $a_1, \dots, a_n \in \{0, 1\}$ such that $x = p(y_0 + \frac{a_1}{2}e_1 + \dots + \frac{a_n}{2}e_n)$. If $y_0 = (y_1^0, \dots, y_n^0)$ and $x = (x_1, \dots, x_n)$, then

$$x_1 + \dots + x_n = y_1^0 + \dots + y_n^0 + \frac{a_1}{2} + \dots + \frac{a_n}{2} \pmod{1}.$$

Define $a = y_1^0 + \dots + y_n^0$ and observe that

$$\frac{a_1}{2} + \dots + \frac{a_n}{2} \pmod{1} = \begin{cases} 1/2 & \text{if } \#\{a_i : a_i = 1\} \text{ is odd} \\ 0 & \text{if } \#\{a_i : a_i = 1\} \text{ is even.} \end{cases}$$

From this we conclude that the map

$$h^{-1}(y) \ni (x_1, \dots, x_n) \mapsto x_1 + \dots + x_n \pmod{1},$$

has 2 possible values: a and $a + 1/2$. In particular, each one is achieved by $2^m/2$ points of $h^{-1}(y)$. In order to understand the distribution of the preimages along the fiber we only need to study two maps, ψ_a and $\psi_{a+1/2}$.

Let us call $I \subset \mathbb{T}^1$ the interval where $\frac{\partial}{\partial t}\psi_a = \lambda$. By construction $|\psi_a(I)| = 1/2$ and therefore $\psi_a(I) \cap \psi_{a+1/2}(I) = \emptyset$. This means that, given $s \in \mathbb{T}^1$, unless s lies on the boundary of $\psi_a(I)$, then either $s \in \psi_a(I)$ or $s \in \psi_{a+1/2}(I)$. If $s \in \psi_a(I)$, then there exists $t_0 \in \psi_a^{-1}(s)$ such that $\frac{\partial}{\partial t}\psi_a(t_0) = \lambda$ and for the remaining $k - 1$ points in $t \in \psi_a^{-1}(s)$ we have $\frac{\partial}{\partial t}\psi_a(t) = \eta$. On the other hand, since $s \notin \psi_{a+1/2}(I)$, we have $\frac{\partial}{\partial t}\psi_{a+1/2}(t) = \eta$ for all $t \in \psi_{a+1/2}^{-1}(s)$.

Note that, since f is a skew-product, we have $J(f, (x, t)) = J(h, x)J(g_x, t)$. Consequently, on the full volume set where $J(f, (x, t))$ is well defined, it can attain one out of two possible

values, $2^m\lambda$ or $2^m\eta$. We now put everything together. By Equation 6, to prove that f is conservative is equivalent to check that

$$\frac{2^n}{2} \left(\frac{1}{2^n\lambda} + (k-1)\frac{1}{2^n\eta} \right) + \frac{2^n}{2} k \frac{1}{2^n\eta} = 1.$$

This is can be simplified to

$$\frac{1}{\lambda} + \frac{2k-1}{\eta} = 2,$$

and replacing η by its value $\frac{(2k-1)\lambda}{2\lambda-1}$ we verify the previous equation and therefore the map f is conservative. \square

3 Fundamental Group of Invariant Regions

Through out this section $f : \mathbb{T}^n \rightarrow \mathbb{T}^n$ will be a volume-preserving endomorphism.

Definition 1: We say that an open subset $U \subset \mathbb{T}^n$ is an invariant region for $f : \mathbb{T}^n \rightarrow \mathbb{T}^n$ if $f^{-1}(U) = U$.

The motivation for this definition is the observation that if U is an invariant region for f , then U together with the restriction of f to U is itself a covering space.

Proposition 3.1: If $f : \mathbb{T}^n \rightarrow \mathbb{T}^n$ is a conservative endomorphism, then the following are equivalent:

- f is not transitive,
- there exist $U, V \subset \mathbb{T}^n$ invariant regions for \mathbb{T}^n , such that U is equal to the interior of $\mathbb{T}^n - V$.

For a proof of this proposition check further Proposition 3.2 in [A]. Our objective now is to have a more comfortable framework. This means to suppose that U is connected.

Lemma 3.2: Let $f : \mathbb{T}^n \rightarrow \mathbb{T}^n$ be a conservative non-invertible endomorphism and suppose that U is an invariant region. If U_0 is a connected component of U , then there exists $m \geq 1$ such that U_0 is an invariant region for f^m .

See Lemma 3.9 in [A] for a proof.

The proof of Theorem 1 will be by contradiction. Suppose that f is not transitive. Then, by Proposition 3.1, there are disjoint invariant regions U and V for f . By Lemma 3.2 each connected component of U and V is periodic. This means that there exist $m_1, m_2 \geq 1$, and connected components U_0 and V_0 of U and V respectively, such that U_0 is an invariant region for f^{m_1} and V_0 is an invariant region for f^{m_2} . In particular, taking $m = m_1 m_2$, we have that both U_0 and V_0 are invariant regions for f^m . Since we are assuming that f is not transitive, neither is f^m . Now, clearly the hypotheses in Theorem 1 also hold for f^m . Therefore it suffices to consider the case in which both U and V are connected.

Lemma 3.3: Let $f : \mathbb{T}^n \rightarrow \mathbb{T}^n$ be a conservative non-invertible endomorphism and suppose that U is an invariant region. If $i : U \rightarrow \mathbb{T}^n$ is the inclusion and $i_\# : \pi_1(U) \rightarrow \pi_1(\mathbb{T}^n)$ is the group morphism induced by i , then $i_\#$ is not trivial.

See Lemma 3.6 in [A] for a proof. From now on, we will assume that if U is an invariant region, it is also connected.

Let us set the following notation. Given $f : \mathbb{T}^n \rightarrow \mathbb{T}^n$ and U an invariant region, take $i : U \rightarrow \mathbb{T}^n$ to be the inclusion. Let $p : \mathbb{R}^n \rightarrow \mathbb{T}^n$ be the natural projection. A lift of f is a homeomorphism $\hat{f} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $f \circ p = p \circ \hat{f}$. We write $\hat{U} = p^{-1}(U)$. The composition of a lift of f with a translation by a vector in \mathbb{Z}^2 is again a lift of f . Consequently, we can (and do) choose a lift \hat{f} of f such that \hat{U} is invariant for \hat{f} .

Lemma 3.4: Let $f : \mathbb{T}^n \rightarrow \mathbb{T}^n$ be an endomorphism and U an invariant region. Take $i : U \rightarrow \mathbb{T}^n$, $p : \mathbb{R}^n \rightarrow \mathbb{T}^n$, $\hat{U} \subset \mathbb{R}^n$ and $\hat{f} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ as before. Then

1. the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi_1(\hat{U}) & \xrightarrow{p_{\#}} & \pi_1(U) & \xrightarrow{i_{\#}} & \pi_1(\mathbb{T}^n) \\ & & \downarrow \hat{f}_{\#} & & \downarrow (f|_U)_{\#} & & \downarrow f_{\#} \\ 0 & \longrightarrow & \pi_1(\hat{U}) & \xrightarrow{p_{\#}} & \pi_1(U) & \xrightarrow{i_{\#}} & \pi_1(\mathbb{T}^n) \end{array}$$

is commutative, and

2. the sequence $0 \rightarrow \pi_1(\hat{U}) \xrightarrow{p_{\#}} \pi_1(U) \xrightarrow{i_{\#}} \pi_1(\mathbb{T}^n)$ is exact.

Proof. The commutativity of the first square follows from the fact that $p \circ \hat{f} = f \circ p$. The commutativity of the second square follows from the fact that $i : U \rightarrow \mathbb{T}^n$ is the inclusion.

Let us prove the exactness. Observe that the injectivity of $p_{\#}$ holds because p is a covering map. In order to prove that $\text{Ker}(i_{\#}) = \text{Im}(p_{\#})$, fix a point $\hat{x} \in \hat{U}$ and $x = p(\hat{x})$. Given $\gamma : [0, 1] \rightarrow U$ a continuous curve such that $\gamma(0) = \gamma(1) = x$ observe that $[\gamma] \in \text{Ker}(i_{\#})$ if γ is homotopic to the constant curve x in \mathbb{T}^n . This happens if and only if the lift $\hat{\gamma}$ of γ on \hat{x} verifies $\hat{\gamma}(0) = \hat{\gamma}(1)$. Therefore $\hat{\gamma}$ is a closed curve in \hat{U} which represents an element of $\pi_1(\hat{U})$, and $p_{\#}([\hat{\gamma}]) = [\gamma]$. \square

Remark 3.5: In the previous situation, $i_{\#} : \pi_1(U) \rightarrow \pi_1(\mathbb{T}^n)$ might not be surjective. However, if we replace $\pi_1(\mathbb{T}^n)$ with $i_{\#}(\pi_1(U))$, we obtain that the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi_1(\hat{U}) & \xrightarrow{p_{\#}} & \pi_1(U) & \xrightarrow{i_{\#}} & i_{\#}(\pi_1(U)) \longrightarrow 0 \\ & & \downarrow \hat{f}_{\#} & & \downarrow (f|_U)_{\#} & & \downarrow f_{\#} \\ 0 & \longrightarrow & \pi_1(\hat{U}) & \xrightarrow{p_{\#}} & \pi_1(U) & \xrightarrow{i_{\#}} & i_{\#}(\pi_1(U)) \longrightarrow 0 \end{array}$$

is commutative and the sequence $0 \rightarrow \pi_1(\hat{U}) \xrightarrow{p_{\#}} \pi_1(U) \xrightarrow{i_{\#}} i_{\#}(\pi_1(U)) \rightarrow 0$ is exact. Therefore, $i_{\#}(\pi_1(U))$ is isomorphic to the quotient group $\pi_1(U)/\text{Im}(p_{\#})$.

Definition 2: Given a group morphism $\phi : H \rightarrow G$ we define the degree of ϕ by $\deg(\phi) = [G : \phi(H)]$, this is the number of elements in the quotient $G/\phi(H)$.

Remark 3.6: In the previous definition, if $H = G = \mathbb{Z}^n$, then ϕ can be represented by a matrix $A_{\phi} \in M_n(\mathbb{Z})$. In such case, $\deg(\phi) = |\det(A_{\phi})|$.

We recall a classical result from the theory of covering spaces.

Theorem 3.7: Let X and Y be path connected topological spaces and $g : X \rightarrow Y$ a covering map. Then, the number of sheets of g is equal to $\deg(g_{\#})$, where $g_{\#} : \pi_1(X) \rightarrow \pi_1(Y)$ is the group morphism induced by g .

A proof of this result can be found in [H].

The following lemma, in combination with Theorem 3.7 and Lemma 3.4, will be the main ingredient in the proof of Theorem 1. It is a purely algebraic result:

Lemma 3.8: Let H, G and K be groups, and let $\alpha : H \rightarrow G$, $\beta : G \rightarrow K$, $\phi : H \rightarrow H$, $\psi : G \rightarrow G$ and $\nu : K \rightarrow K$ be group morphisms such that:

- ϕ is an isomorphism.
- the sequence $H \xrightarrow{\alpha} G \xrightarrow{\beta} K \rightarrow 0$ is exact.
- the diagram

$$\begin{array}{ccccccc} H & \xrightarrow{\alpha} & G & \xrightarrow{\beta} & K & \longrightarrow & 0 \\ \downarrow \phi & & \downarrow \psi & & \downarrow \nu & & \\ H & \xrightarrow{\alpha} & G & \xrightarrow{\beta} & K & \longrightarrow & 0 \end{array}$$

is commutative.

Then, $\deg(\psi) = \deg(\nu)$.

Proof. Take $N = \text{Im}(\alpha) = \text{Ker}(\beta) \triangleleft G$. Then, $\psi(N) = \psi(\alpha(H)) = \alpha(\phi(H)) = \alpha(H) = N$ because ϕ is an isomorphism. This allow us to define the morphism $\tilde{\psi} : G/N \rightarrow G/N$ by $\tilde{\psi}(gN) = \psi(g)N$. Take also $\tilde{\beta} : G/N \rightarrow K$ defined by $\tilde{\beta}(gN) = \beta(g)$. Since β is surjective and $N = \text{Ker}(\beta)$, $\tilde{\beta}$ is an isomorphism. Since $\nu \circ \beta = \beta \circ \psi$, we have $\nu \circ \tilde{\beta} = \tilde{\beta} \circ \tilde{\psi}$. This means that the following diagram is commutative:

$$\begin{array}{ccc} G/N & \xrightarrow{\tilde{\beta}} & K \\ \downarrow \tilde{\psi} & & \downarrow \nu \\ G/N & \xrightarrow{\tilde{\beta}} & K \end{array}$$

Since $\tilde{\beta}$ is an isomorphism, we have $\deg(\nu) = \deg(\tilde{\psi})$. It remains then to prove that $\deg(\psi) = \deg(\tilde{\psi})$. This is, we need to see that $[G : \psi(G)] = [G/N : \tilde{\psi}(G/N)]$. Since $N = \psi(N)$, we have that $N \triangleleft \psi(G) < G$. Now, $\tilde{\psi}(G/N) = \{\psi(g)N : g \in G\} = \psi(G)/N$, where the first equality comes from the definition of $\tilde{\psi}$ and the second holds because $N < \psi(G)$. So our problem is reduced to prove that $[G : \psi(G)] = [G/N : \psi(G)/N]$. In order to do this, we define the map $\eta : G/\psi(G) \rightarrow (G/N)/(\psi(G)/N)$ by $\eta(g\psi(G)) = (gN)(\psi(G)/N)$. It is well defined and bijective. Hence $\deg(\psi) = \deg(\tilde{\psi})$. \square

Let us fix now a convenient notation. Given a subset $B \subset \mathbb{R}^n$ we define $\langle B \rangle \subset \mathbb{R}^n$ as the subspace induced by B .

The following is the main lemma of this paper:

Lemma 3.9: Let $f : \mathbb{T}^n \rightarrow \mathbb{T}^n$ be a volume-preserving endomorphism and U an invariant region. If $S = \langle i_{\#}(\pi_1(U)) \rangle$, then $|\det(A_{f|S})| = |\det(A_f)|$.

Proof. We observed in Remark 3.5 that the diagram

$$\begin{array}{ccccccc} \pi_1(\hat{U}) & \xrightarrow{p_{\#}} & \pi_1(U) & \xrightarrow{i_{\#}} & i_{\#}(\pi_1(U)) & \longrightarrow & 0 \\ \downarrow \hat{f}_{\#} & & \downarrow (f|_U)_{\#} & & \downarrow (f_{\#})|_{i_{\#}(\pi_1(U))} & & \\ \pi_1(\hat{U}) & \xrightarrow{p_{\#}} & \pi_1(U) & \xrightarrow{i_{\#}} & i_{\#}(\pi_1(U)) & \longrightarrow & 0 \end{array}$$

is commutative and the sequence $\pi_1(\hat{U}) \xrightarrow{p_{\#}} \pi_1(U) \xrightarrow{i_{\#}} i_{\#}(\pi_1(U)) \rightarrow 0$ is exact. Since \hat{f} is a homeomorphism, $\hat{f}_{\#}$ is an isomorphism and we can apply Lemma 3.8. We have the following equation:

$$|\det(A_{f|S})| \stackrel{(1)}{=} \deg((f_{\#})_{|i_{\#}(\pi_1(U))}) \stackrel{(2)}{=} \deg((f|_U)_{\#}) \stackrel{(3)}{=} \deg((f|_U)) \stackrel{(4)}{=} \deg(f) \stackrel{(5)}{=} \deg(f_{\#}) \stackrel{(6)}{=} |\det(A_f)|, \quad (7)$$

where (1) and (6) holds by Remark 3.6, (2) holds by Lemma 3.8 and because by Lemma 3.3 $i_{\#}(\pi_1(U))$ is not trivial, (3) and (5) by Theorem 3.7 and (4) because $\deg((f|_U))$ is the number of preimages of any point for the map $f|_U$, since U is an invariant region this number coincides with the number of preimages of f which is $\deg(f)$. \square

Remark 3.10: *If we remove the volume-preserving hypothesis from Lemma 3.9, we obtain that $i_{\#}(\pi_1(U)) = \{0\}$ or $|\det(A_{f|S})| = |\det(A_f)|$.*

4 Invariant subspaces of an Integer Matrix

The objective of this section is to prove the following proposition:

Proposition 4.1: *Given $A \in M_n(\mathbb{Z})$ and $\{0\} \subsetneq S \subsetneq \mathbb{R}^n$ an invariant subspace by A . If $\det(A|_S) \in \mathbb{Z} - \{0\}$, then $\det(A|_S)$ divides $\det(A)$. In particular, $|\det(A|_S)| \leq |\det(A)|$.*

With this in mind, we start by showing that if λ is a rational eigenvalue of an integer matrix, then λ is an integer. Indeed this is a direct consequence of the well known Rational Root Theorem in elementary algebra. We include it for completeness and because it serves as a warm-up for the proof of Proposition 4.1.

Lemma 4.2: *Given $A \in M_n(\mathbb{Z})$ if $\lambda \in \mathbb{Q}$ is an eigenvalue of A , then $\lambda \in \mathbb{Z}$.*

Proof. If χ_A is the characteristic polynomial of A , then all the coefficients of χ_A belong to \mathbb{Z} and moreover χ_A is monic. Suppose that $\chi_A(t) = (-1)^n t^n + \sum_{i=0}^{n-1} a_i t^i$ and take $p, q \in \mathbb{Z}$ coprimes, with $q \neq 0$ such that $\chi_A(\frac{p}{q}) = 0$, then $0 = \frac{p^n}{q^n} + \frac{r}{q^{n-1}}$ for some $r \in \mathbb{Z}$. If $r = 0$, then $p = 0$ and we are done. If $r \neq 0$, then $-qr = p^n$. Since we took q and p coprime, the later equation implies that $q = \pm 1$ and we conclude. \square

The following lemma extends the previous lemma to invariant subspaces.

Lemma 4.3: *Given $A \in M_n(\mathbb{Z})$ and $\{0\} \subsetneq S \subsetneq \mathbb{R}^n$ an invariant subspace by A . If $\det(A|_S) \in \mathbb{Q}$, then $\det(A|_S) \in \mathbb{Z}$.*

Proof. Given $1 \leq m \leq n$ we define the m exterior power of \mathbb{R}^n by $V_m = \overbrace{\mathbb{R}^n \otimes \cdots \otimes \mathbb{R}^n}^{m \text{ times}}$. In V_m we define the linear map $A_m : V_m \rightarrow V_m$ by $A_m(v_1 \otimes \cdots \otimes v_m) = A(v_1) \otimes \cdots \otimes A(v_m)$. If $\{e_1, \dots, e_n\}$ is the canonical basis in \mathbb{R}^n , then $\{e_{i_1} \otimes \cdots \otimes e_{i_m} : 1 \leq i_1 < \cdots < i_m \leq n\}$ is a basis for V_m . Each A_m can be represented by a matrix with respect to this basis. Since $A \in M_n(\mathbb{Z})$, these matrices have integer coefficients. Notice that if $S \subset \mathbb{R}^n$ is a subspace invariant under A , then $\det(A|_S)$ is an eigenvalue of A_m where $m = \dim(S)$. Hence, applying Lemma 4.2 to the (integer) matrix of A_m , we conclude that $\det(A|_S)$ is an integer. \square

The next lemma will be the final piece to prove Proposition 4.1.

Lemma 4.4: *Given $A \in M_n(\mathbb{Z})$ and $\{0\} \subsetneq S \subsetneq \mathbb{R}^n$ an invariant subspace by A . Then, there exists W an invariant subspace by A such that $\det(A) = \det(A|_S) \det(A|_W)$.*

Proof. Let us suppose that A is diagonalizable. In that case there is a basis of \mathbb{R}^n consisting of eigenvectors $\{v_1, \dots, v_n\}$ of A . Since S is invariant under A , there exist $1 \leq i_1 < \cdots < i_m \leq n$ such that $S = \langle \{v_{i_1}, \dots, v_{i_m}\} \rangle$, where $m = \dim(S)$. Therefore, if we take $W = \langle \{v_i : i \neq i_j \forall j = 1, \dots, m\} \rangle$, then W is invariant under A . Observe that $\det(A|_S) = \prod_{j=1}^m \lambda_{i_j}$ and $\det(A|_W) = \prod_{i \notin \{i_1, \dots, i_m\}} \lambda_i$, where λ_i is the eigenvalue associated to v_i for all $1 \leq i \leq n$. Since $\det(A) = \prod_{i=1}^n \lambda_i$, we have $\det(A) = \det(A|_S) \det(A|_W)$.

We are going to address the case when A is not diagonalizable and it has no complex eigenvalues. The other case will be discussed later. We will now take the real Jordan form associated to A . Let us briefly recall what this is. If A is diagonalizable, it means that there exist a diagonal matrix D associated to A and a basis \mathcal{B} (formed by eigenvectors) such that the linear map associated to A is represented by D in the basis \mathcal{B} . When A is not diagonalizable, we have an almost diagonal matrix J associated to A and a basis \mathcal{B} such that the linear map associated to A is represented by J in the basis \mathcal{B} .

With J the real Jordan form of A we are going to decompose our invariant subspace S in small invariant subspaces S_l , where each one will be a subspace associated to a Jordan

Block J_l . For each S_l we are going to build an invariant subspace W_l and then $W = \oplus_l W_l$ will verify the desired equation.

Given a Jordan block J_l we consider \mathcal{B}_l the elements of the Jordan basis \mathcal{B} associated to J_l . That is, \mathcal{B}_l is the set $\{v_1^l, \dots, v_{k_l}^l\} \subset \mathcal{B}$ such that $A(v_j^l) = \lambda_l v_j^l + v_{j+1}^l$ if $1 \leq j < k_l$ and $A(v_{k_l}^l) = \lambda_l v_{k_l}^l$ where λ_l is the eigenvalue associated to J_l . Consider the subspace V_l induced by \mathcal{B}_l . Note that $S_l = V_l \cap S$ is invariant under A and therefore is going to be the subspace induced by $\{v_{m_l}^l, \dots, v_{k_l}^l\}$ where $k_l - m_l = \dim(S_l)$. Observe that we cannot build W as before because the induced space by $\{v_1^l, \dots, v_{m_l-1}^l\}$ is not invariant. However, if we define W_l by $W_l = \langle \{v_{k_l-m_l}^l, \dots, v_{k_l}^l\} \rangle$, then it is invariant and the equation $\det(A|_{V_l}) = \det(A|_{S_l}) \det(A|_{W_l})$ holds. Now if we define $W = \oplus_l W_l$, we have that W is invariant by A and the equation $\det(A) = \det(A|_S) \det(A|_W)$ also holds.

When there are complex eigenvalues, since the characteristic polynomial of A has real coefficients, these necessarily come in pairs of complex conjugates. For each of this pairs corresponds a two dimensional subspace on which A acts as a composition of a rotation with a homothecy. Its Jordan block is of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ in \mathbb{R}^2 . From this, if we take \mathcal{B}_l to be the elements of the Jordan basis \mathcal{B} associated to the Jordan block J_l where the eigenvalue λ_l is complex, we have the following:

$$\mathcal{B}_l = \{v_1^{l,1}, v_1^{l,2}, \dots, v_{k_l}^{l,1}, v_{k_l}^{l,2}\},$$

$$A(v_j^{l,1}) = a_l v_j^{l,1} - b_l v_j^{l,2} + v_{j+1}^{l,1} \text{ if } 1 \leq j < k_l,$$

$$A(v_j^{l,2}) = b_l v_j^{l,1} + a_l v_j^{l,2} + v_{j+1}^{l,2} \text{ if } 1 \leq j < k_l,$$

$$A(v_{k_l}^{l,1}) = a_l v_{k_l}^{l,1} - b_l v_{k_l}^{l,2},$$

and

$$A(v_{k_l}^{l,2}) = b_l v_{k_l}^{l,1} + a_l v_{k_l}^{l,2}.$$

In this case, if V_l is the subspace induced by \mathcal{B}_l and $S_l = S \cap V_l$, we have that S_l is the subspace induced by $\{v_{m_l}^{l,1}, v_{m_l}^{l,2}, \dots, v_{k_l}^{l,1}, v_{k_l}^{l,2}\}$ where $2(k_l - m_l) = \dim(S_l)$. We then build analogously W_l and W .

□

Proof of Proposition 4.1. Given $\{0\} \subsetneq S \subsetneq \mathbb{R}^n$ such that $\det(A|_S) \in \mathbb{Z}$, by the previous Lemma take W invariant by A which verifies $\det(A|_S) \det(A|_W) = \det(A)$. Since $\det(A|_S) \in \mathbb{Z}$ and $\det(A) \in \mathbb{Z}$, then $\det(A|_W) \in \mathbb{Q}$. By Lemma 4.3, $\det(A|_W) \in \mathbb{Z}$. □

5 Skew-Products of codimension 1

The main objective of this section is to prove Theorem 1 and Corollaries 1 and 2.

Let us define $\hat{r}_1 : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ and $r_1 : \mathbb{T}^n \rightarrow \mathbb{T}^{n-1}$ by

$$\hat{r}_1(x_1, \dots, x_n) = (x_1, \dots, x_{n-1}) \quad \forall (x_1, \dots, x_n) \in \mathbb{R}^n,$$

and

$$r_1(x_1, \dots, x_n) = (x_1, \dots, x_{n-1}) \quad \forall (x_1, \dots, x_n) \in \mathbb{T}^n.$$

The skew-product structure of f implies that $r_1 \circ f = h \circ r_1$. In particular, we can take lifts $\hat{f} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\hat{h} : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1}$ of f and h such that $\hat{r}_1 \circ \hat{f} = \hat{h} \circ \hat{r}_1$.

The following two properties verified by a lift of f and the linear map of f come from classical arguments (check further [W]):

Proposition 5.1: *Let $f : \mathbb{T}^n \rightarrow \mathbb{T}^n$ be a continuous map, $\hat{f} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a lift of f , $f_\# : \mathbb{Z}^n \rightarrow \mathbb{Z}^n$ the induced action by f in the fundamental group of \mathbb{T}^n and $A_f \in M_n(\mathbb{Z})$ the matrix that represents $f_\#$. Then, we have the following equations:*

$$\hat{f}(x + v) = \hat{f}(x) + f_\#(v) \quad \forall x \in \mathbb{R}^n, \quad \forall v \in \mathbb{Z}^n, \quad (8)$$

$$\exists L_0 > 0 \text{ such that } d(\hat{f}(x), A_f(x)) \leq L_0 \quad \forall x \in \mathbb{R}^n. \quad (9)$$

For the sake of completeness we give a proof of these statements.

Proof. Let $p : \mathbb{R}^n \rightarrow \mathbb{T}^n$ be the natural projection and define $\hat{\alpha} : [0, 1] \rightarrow \mathbb{R}^n$ and $\alpha : [0, 1] \rightarrow \mathbb{T}^n$ by $\hat{\alpha}(t) = x + tv$ and $\alpha(t) = p(\hat{\alpha}(t))$. In particular, $\hat{f} \circ \hat{\alpha}$ is curve which starts at $\hat{f}(x)$ and ends at $\hat{f}(x + v)$. On the other hand, by definition, α is a loop and $[\alpha] = v$. Now $p \circ \hat{f} \circ \hat{\alpha} = f \circ p \circ \hat{\alpha} = f \circ \alpha$ and therefore $\hat{f} \circ \hat{\alpha}$ is a lift of the curve $f \circ \alpha$. By definition, $f_\#(v) = f_\#([\alpha]) = [f \circ \alpha]$ and then $\hat{f} \circ \hat{\alpha}$ is a lift which begins at $\hat{f}(x)$ and ends at $\hat{f}(x) + f_\#(v)$ obtaining that $\hat{f}(x + v) = \hat{f}(x) + f_\#(v)$.

For the second equation define $\lfloor x \rfloor \in \mathbb{Z}^n$ such that $x - \lfloor x \rfloor \in [0, 1)^n$. Using Equation 8, we have:

$$\hat{f}(x) = \hat{f}(x - \lfloor x \rfloor + \lfloor x \rfloor) = \hat{f}(x - \lfloor x \rfloor) + f_\#(\lfloor x \rfloor),$$

then

$$\hat{f}(x) - A_f(x) = \hat{f}(x - \lfloor x \rfloor) + f_\#(\lfloor x \rfloor) - A_f(x - \lfloor x \rfloor) - A_f(\lfloor x \rfloor).$$

Since $f_\#(\lfloor x \rfloor) = A_f(\lfloor x \rfloor)$, we obtain $\hat{f}(x) - A_f(x) = \hat{f}(x - \lfloor x \rfloor) - A_f(x - \lfloor x \rfloor)$. This implies that $\text{Im}(\hat{f} - A_f) = \text{Im}((\hat{f} - A_f)|_{[0, 1]^n})$. Using the compactness of $[0, 1]^n$, we conclude the proposition. \square

We return to the skew-product structure with the following construction: As we said before $\pm \deg(g)$ is an eigenvalue of A_f and $e_n = (0, \dots, 0, 1)$ is the eigenvector associated to it (the eigenvalue is $\deg(g)$ if f preserves the orientation on the fiber and $-\deg(g)$ if f reverses it). Let $\{v_1, \dots, v_{n-1}, e_n\}$ be a Jordan basis for A_f . If J_n is the Jordan block associated to e_n and if $\dim(J_n) = 1$, then $P_0 = \langle \{v_1, \dots, v_{n-1}\} \rangle$ is a hyperplane invariant by A_f , transverse to e_n . In particular, the transverse condition implies that $\hat{r}_1|_{P_0}$ is a linear isomorphism. This is the only place where we use the hypothesis $\dim(J_n) = 1$. It guarantees the existence of P_0 .

Remark 5.2: *Let λ_n be the eigenvalue of A_f associated to the eigenvector e_n . Observe that if $\lambda_n < 0$, then $\lambda_n^2 > 0$ is the eigenvalue associated to e_n under the map $A_f^2 = A_{f^2}$. If f is not transitive, neither is f^2 . We may therefore assume in what follows that f preserves the orientation in the fibers.*

Given two parallel hyperplanes $P_1, P_2 \subset \mathbb{R}^n$ we call $[P_1, P_2] \subset \mathbb{R}^n$ the connected set which has $P_1 \cup P_2$ as its boundary and for which $\mathbb{R}^n \cap [P_1, P_2]^c$ has two connected components.

Lemma 5.3: *There exists $k_1 \leq k_2 \in \mathbb{Z}$ such that*

$$\hat{f}^{-1}([P_0 + k_1 e_n, P_0 + k_2 e_n]) \subset [P_0 + k_1 e_n, P_0 + k_2 e_n].$$

Proof. By Proposition 5.1, take $L_0 > 0$ such that $d(\hat{f}(x), A_f(x)) \leq L_0 \forall x \in \mathbb{R}^n$. Since P_0 is invariant for A_f , we have that

$$\hat{f}(P_0) \subset [P_0 - L_0 e_n, P_0 + L_0 e_n].$$

Combining this and the Equation 8 from Proposition 5.1, we obtain

$$\hat{f}(P_0 + k e_n) \subset [P_0 + (\deg(g)k - L_0)e_n, P_0 + (\deg(g)k + L_0)e_n].$$

Take $k_1, k_2 \in \mathbb{Z}$ such that $\deg(g)k_2 - L_0 \geq k_2$ and $\deg(g)k_1 + L_0 \leq k_1$. We can find such k_1 and k_2 because $\deg(g) \geq 2$. In particular, $k_1 \leq 0 \leq k_2$. For these k_1 and k_2 we have that

$$\hat{f}([P_0 + k_1 e_n, P_0 + k_2 e_n]) \supset [P_0 + k_1 e_n, P_0 + k_2 e_n].$$

Since \hat{f} is a homeomorphism, if we apply \hat{f}^{-1} to the previous equation, we conclude the lemma. \square

We define the set S_0 by

$$S_0 = \bigcap_{k \in \mathbb{N}} \hat{f}^{-k}([P_0 + k_1 e_n, P_0 + k_2 e_n]).$$

Let us define $r_0 : S_0 \rightarrow \mathbb{R}^{n-1}$ by $r_0(s) = \hat{r}_1(s) \forall s \in S_0$.

Given a continuous curve $\alpha : [0, 1] \rightarrow \mathbb{R}^n$ such that $\alpha(1) = \alpha(0) + v$ with $v \in \mathbb{Z}^n$ we define its periodic continuation as $\alpha_\infty : \mathbb{R} \rightarrow \mathbb{R}^n$ by $\alpha_\infty(t) = v[t] + \alpha(t - [t])$, where $[t]$ is the integer part of t .

Lemma 5.4: *The set S_0 verifies the following:*

1. $\hat{f}(S_0) = S_0$.
2. $r_0(S_0) = \mathbb{R}^{n-1}$.
3. $\mathbb{R}^n \cap S_0^c$ has two connected components.
4. Given $\alpha : [0, 1] \rightarrow \mathbb{R}^n$ such that $\alpha(1) = \alpha(0) + v$ with v transverse to P_0 , then $S_0 \cap \text{Im}(\alpha_\infty) \neq \emptyset$.
5. $r_0^{-1}(x)$ is a connected set for every $x \in \mathbb{R}^{n-1}$ and is therefore either a point or an interval.
6. r_0 is a semi-conjugacy between $\hat{f}|_{S_0}$ and \hat{h} . This is $\hat{h} \circ r_0 = r_0 \circ \hat{f}|_{S_0}$.

Proof. 1. By definition of S_0 .

2. Given $x \in \mathbb{R}^{n-1}$ we have that

$$\emptyset \neq \bigcap_{k \in \mathbb{N}} \hat{f}^k(\{\hat{h}^{-k}(x)\} \times [k_1, k_2]) \subset S_0 \cap \hat{r}_1^{-1}(x).$$

3. $\mathbb{R}^n \cap \hat{f}^{-k}([P_0 + k_1 e_n, P_0 + k_2 e_n])^c$ has two connected components because \hat{f} is a homeomorphism, and $\mathbb{R}^n \cap [P_0 + k_1 e_n, P_0 + k_2 e_n]^c$ has two connected components. Then, this property is verified by S_0 .

4. Given such a curve α , α_∞ intersects both connected components of $\mathbb{R}^n \cap [P_0 + k_1 e_n, P_0 + k_2 e_n]^c$. Therefore α intersects both connected components of $\mathbb{R}^n \cap S_0^c$. Since α_∞ is continuous, it intersects S_0 in some point.
5. If two points project to the same point, then the dynamics of both remain between the two hyperplanes. Since \hat{f} is a skew-product, the whole segment remains between the two planes.
6. This is because \hat{r}_1 is a semi-conjugacy between \hat{f} and \hat{h} .

□

Lemma 5.5: *There exists $S \subset \mathbb{R}^n$ which verifies:*

1. $\hat{f}(S) = S$.
2. If we define $r : S \rightarrow \mathbb{R}^{n-1}$ by $r = \hat{r}_1|_S$, then $r(S) = \mathbb{R}^{n-1}$.
3. Given $\alpha : [0, 1] \rightarrow \mathbb{R}^n$ such that $\alpha(1) = \alpha(0) + v$ with v transverse to P_0 , then $S \cap \text{Im}(\alpha_\infty) \neq \emptyset$.
4. $r^{-1}(x)$ is a connected set $\forall x \in \mathbb{R}^{n-1}$, therefore either a point or an interval.
5. r is a semi-conjugacy between $\hat{f}|_S$ and \hat{h} . This is $\hat{h} \circ r = r \circ \hat{f}|_S$.
6. The interior of S is empty.

Proof. The set S_0 in Proposition 5.4 has all the above properties except possibly for the last one. If it does, we take $S = S_0$ and we are done. If this is not the case, for each $x \in \mathbb{R}^{n-1}$ we define $a(x) \leq b(x) \in \mathbb{R}$ such that $r_0^{-1}(x) = \{x\} \times [a(x), b(x)]$. We build now the following two sets

$$\min(S_0) = \left(\bigcup_{x \in r_0(\text{Int}(S_0))} \{(x, a(x))\} \right) \cup \left(\bigcup_{x \notin r_0(\text{Int}(S_0))} r_0^{-1}(x) \right),$$

and

$$\max(S_0) = \left(\bigcup_{x \in r_0(\text{Int}(S_0))} \{(x, b(x))\} \right) \cup \left(\bigcup_{x \notin r_0(\text{Int}(S_0))} r_0^{-1}(x) \right).$$

These sets are well defined by item 2 and 5 in Lemma 5.4. In particular, the boundary of S_0 verifies

$$\partial S_0 = \min(S_0) \cup \max(S_0).$$

Since f preserves the orientation in the fibers, we have that $\hat{f}(\max(S_0)) = \max(S_0)$ and $\hat{f}(\min(S_0)) = \min(S_0)$. Now $\max(S_0)$ and $\min(S_0)$ verify all the properties of Lemma 5.4 and $\text{int}(\max(S_0)) = \emptyset$. We define then S as $\max(S_0)$. □

We are now in condition to prove Theorem 1.

Proof of Theorem 1. Let us assume that f is not transitive. Let U and V be as in Proposition 3.1 and write $\hat{U} = p^{-1}(U)$ and $\hat{V} = p^{-1}(V)$. Let $i : U \rightarrow \mathbb{T}^n$ be the inclusion and denote by P_1 the subspace $\langle i_*(\pi_1(U)) \rangle$ of \mathbb{R}^n . Suppose that $i_*(\pi_1(U))$ has no element transverse to P_0 , this means P_1 is an A_f invariant subspace of P_0 . Since $\hat{r}_1|_{P_0} : P_0 \rightarrow \mathbb{R}^{n-1}$ is a linear isomorphism which conjugates $A_{f|P_0}$ and A_h , then $\hat{r}_1(P_1)$ is an invariant subspace of A_h and $|\det(A_h|_{\hat{r}_1(P_1)})| = |\det(A_f|_{P_1})|$. By Lemma 3.9, $|\det(A_f|_{P_1})| = |\det(A_f)|$, and since

$$A_f = \begin{pmatrix} A_h & 0 \\ * & \deg(g) \end{pmatrix},$$

we have $|\det(A_f)| = \deg(g)|\det(A_h)|$. Therefore $|\det(A_h|_{\hat{r}_1(P_1)})| = \deg(g)|\det(A_h)|$. Since $\deg(g) \geq 2$, we have $|\det(A_h|_{\hat{r}_1(P_1)})| > |\det(A_h)|$ which contradicts Proposition 4.1.

We have proved that $i_*(\pi_1(U))$ has an element transverse to P_0 . Therefore there exists $\alpha : [0, 1] \rightarrow \hat{U}$ such that $\alpha(1) = \alpha(0) + v$ with v transverse to P_0 . By property 3 in Lemma 5.5, $\hat{U} \cap S \neq \emptyset$. Analogously $\hat{V} \cap S \neq \emptyset$. Let us call these intersections U_S and V_S . Since h is transitive, $r(U_S)$ and $r(V_S)$ are open and dense in \mathbb{R}^{n-1} . Take $W = \text{int}(r(U_S) \cap r(V_S)) \neq \emptyset$. Since $\text{int}(S) = \emptyset$, there exists $w \in W$ such that $r^{-1}(w)$ is a point. Such point belongs to $\hat{U} \cap \hat{V}$ which is a contradiction. \square

Let us see why Theorem 1 implies Corollary 1.

Lemma 5.6: *If $f : \mathbb{T}^n \rightarrow \mathbb{T}^n$ is a skew-product endomorphism of the form $f = (h, g)$ such that $|\deg(h)| < |\deg(g)|$, then $\dim(J_n) = 1$.*

Proof. By a simple computation we have:

$$A_f = \begin{pmatrix} A_h & 0 \\ * & \pm \deg(g) \end{pmatrix}.$$

If χ_{A_f} and χ_{A_h} are the characteristic polynomials of A_f and A_h respectively, then $\chi_{A_f}(t) = -\chi_{A_h}(t)(t - \pm \deg(g))$. This implies that the eigenvalues of A_f are $\pm \deg(g)$ and the eigenvalues of A_h . By Proposition 4.1, $\pm \deg(g)$ can not be an eigenvalue of A_h and therefore $\dim(J_n) = 1$. \square

Analogously, let us see why Theorem 1 implies Corollary 2.

Lemma 5.7: *If $h : \mathbb{T}^n \rightarrow \mathbb{T}^n$ is an endomorphism such that $|A_h v| > |\deg(g)||v| \forall v \in \mathbb{R}^n - \{0\}$, then $\dim(J_n) = 1$.*

Proof. By the arguments of the previous lemma, we just need to show that $\pm \deg(g)$ is not an eigenvalue of A_h . If it were, then there would exist $v \in \mathbb{R}^n - \{0\}$ such that $A_h v = \pm \deg(g)v$. This contradicts our hypothesis. \square

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